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Fermion systems and the Moyal formulation of quantum mechanics

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Abstract. Wigner functions of permutation operators are obtained and serve as a basis for a phase-space formulation of quantum $\frac{1}{2}$ -spin identical fermions. We prove a version of the Pauli principle for this type of particle.

1. Introduction

The aim of the present paper is to discuss some aspects of the theory of $\frac{1}{2}$ -spin identical particles in the context of the Moyal formulation of quantum mechanics [1]. This formalism treats quantum states and observables as functions on a suitable phase space. The correspondence with the standard formalism for quantum mechanics is given through the Weyl map [2], which may also be called the Weyl correspondence, and its inverse. An operator on Hilbert space is assigned by the latter a function on phase space—its Wigner function [3].

The phase space suitable for a description of a $\frac{1}{2}$ -spin particle in three-dimensional configuration space is the Cartesian product of \mathbb{R}^6 with the three-dimensional hollow sphere S^2 [4, 5]. \mathbb{R}^6 includes the positions and momenta and S^2 includes the spin. For a system of N identical particles, we can use the Cartesian product of N copies of $\mathbb{R}^6 \times S^2$ as the phase space. If only spin variables are relevant, the phase space is $(S^2)^N$, the product of N times the sphere.

We define the Weyl mapping through the Stratonovich–Weyl kernel [4–6] given by

$$\Omega(u, n) = \Pi(u_1) \otimes \cdots \otimes \Pi(u_N) \otimes \Delta(n_1) \otimes \cdots \otimes \Delta(n_N) \quad (1)$$

where $\Pi(u_i)$, $i = 1, 2, \dots, N$ are the Grossmann–Royer operators defined in [4, 5, 7, 8], $u_i = (q_i, p_i)$ denotes the coordinates of the position and momentum of the i th particle and $\Delta(n_i)$, $i = 1, 2, \dots, N$ are the operators on \mathbb{C}^2 given by

$$\Delta(n_i) = \sum_{r,s=\pm,-} Z_{r,s}(n_i) |s\rangle \langle r| \quad (2)$$

where n_i represents both a point in the i th sphere and its corresponding unit position vector. Here,

$$Z_{r,s}(n) = \sum_{l=0,1} \sqrt{\pi(2l+1)} C(\frac{1}{2}, l, \frac{1}{2}; s, (r-s), r) Y_{l,r-s}(n). \quad (3)$$

$C(\frac{1}{2}, l, \frac{1}{2}; s, (r-s), r)$ are Clebsch–Gordan coefficients and $Y_{l,r-s}$ spherical harmonics. The kets $|+\rangle$ and $|-\rangle$ in (2) are the eigenvectors of the z component of the spin.

The Weyl correspondence maps a function $f(\mathbf{u}, \mathbf{n})$ on $(\mathbb{R}^6 \times S^2)^N$ into the operator on $L^2(\mathbb{R}^{6N}) \otimes \mathbb{C}^{2N}$ given by (henceforth, we will use $\hbar = 1$):

$$W(f) = \frac{1}{(2\pi)^{3N}} \frac{1}{(2\pi)^N} \int_{(\mathbb{R}^6)^N} d\mathbf{u} \int_{(S^2)^N} d\mathbf{n} f(\mathbf{u}, \mathbf{n}) \Omega(\mathbf{u}, \mathbf{n}). \quad (4)$$

This expression admits an inversion formula that gives the Wigner function of the operator $W(f)$ as

$$f(\mathbf{u}, \mathbf{n}) = W^{-1}[W(f)] = \text{tr}[W(f)\Omega(\mathbf{u}, \mathbf{n})]. \quad (5)$$

As customary, we define the twisted product of two functions on $(\mathbb{R}^6 \times S^2)^N$ by $(f \times g)(\mathbf{u}, \mathbf{n}) = W^{-1}[W(f)W(g)]$. This twisted product can be written in integral form as [4–6]

$$(f \times g)(\mathbf{u}, \mathbf{n}) = \int_{(\mathbb{R}^6 \times \mathbb{R}^6)^N} \int_{(S^2 \times S^2)^N} f(\mathbf{v}, \mathbf{m}) g(\mathbf{w}, \mathbf{k}) L(\mathbf{u}, \mathbf{v}, \mathbf{w}) \mathcal{L}(\mathbf{n}, \mathbf{m}, \mathbf{k}) d\mathbf{v} d\mathbf{w} d\mathbf{m} d\mathbf{k} \quad (6)$$

where

(i) $L(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\pi)^{-6N} \exp[2i(\mathbf{u}\mathbf{J}\mathbf{v} + \mathbf{v}\mathbf{J}\mathbf{w} + \mathbf{w}\mathbf{J}\mathbf{u})]$ and $\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. I is the $3N$ -dimensional identity matrix.

(ii) $d\mathbf{m} = dm_1 \dots dm_N$, where dm_i is the Liouville measure on the i th sphere, i.e. $\sin\theta d\theta d\varphi$. The same is true for $d\mathbf{k}$.

(iii) The integral kernel $\mathcal{L}(\mathbf{n}, \mathbf{m}, \mathbf{k})$ is $\mathcal{L}(\mathbf{n}, \mathbf{m}, \mathbf{k}) = \prod_{j=1}^N \mathcal{L}(n_j, m_j, k_j)$, where

$$\mathcal{L}(n_i, m_i, k_i) = \left(\frac{1}{4\pi}\right)^2 \{1 + 3(\mathbf{n}_i \cdot \mathbf{m}_i + \mathbf{m}_i \cdot \mathbf{k}_i + \mathbf{k}_i \cdot \mathbf{n}_i) + 3\sqrt{3}i[\mathbf{n}_i, \mathbf{m}_i, \mathbf{k}_i]\}. \quad (8)$$

If we have a system of particles for which we do not consider either the spin or orbital part, their states and observables as functions on phase space will only depend on \mathbf{u} and \mathbf{n} respectively. Their twisted product can be defined in these cases by omitting the non-relevant part in the kernel and integral in (6) in an obvious manner.

Let H be a Hamiltonian on phase space and $W(H)$ its image by the Weyl map. Let $U(t)$ be the evolution operator produced by this Hamiltonian ($U(t) = e^{-itW(H)}$ if H is time independent). We define the Moyal propagator for H as $\Xi(\mathbf{u}, \mathbf{n}, t) = W^{-1}[U(t)]$. Its Fourier transform with respect to the time variable t gives the spectral projections parametrized by E :

$$\Gamma(\mathbf{u}, \mathbf{n}, E) = \frac{1}{2\pi} \int_{\mathbb{R}} \Xi(\mathbf{u}, \mathbf{n}, t) e^{iEt} dt. \quad (9)$$

The interest of the spectral projections lies in the fact, proven at least for a class of operators including quadratic Hamiltonians [9], that the support on E of $\Gamma(\mathbf{u}, \mathbf{n}, E)$ coincides with the spectrum of the Hamiltonian $W(H)$ and, hence, gives the quantum levels of energy of our system [10].

In the standard formalism of quantum mechanics, the Hilbert space of states of a system of N $\frac{1}{2}$ -spin fermions is the space of the wavefunctions which are antisymmetric under the exchange of two particles. This Hilbert space is obtained if we apply to $\mathcal{H} = L^2(\mathbb{R}^{6N}) \otimes \mathbb{C}^{2N}$ the orthogonal projection given by $P_- = (1/N!) \sum_{\sigma \in S_N} (-1)^\sigma R_\sigma$. S_N is the group of permutations of N elements. For any $\sigma \in S_N$, we define the operator R_σ on \mathcal{H} as $R_\sigma \psi(x_1, x_2, \dots, x_N) = \psi(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)})$, where $\psi(x_1, x_2, \dots, x_N) \in \mathcal{H}$. The label x_i denotes the coordinates of the position (or momentum) and spin of the i th particle and $(-1)^\sigma$ is the parity of σ . We can decompose $R_\sigma = P_\sigma Q_\sigma$, where P_σ and Q_σ produce the permutation σ on the orbital and spin variables respectively. If B is an observable or an state on \mathcal{H} , i.e. in the system before one introduces the notion of identity of the particles, its corresponding operator for the system of N identical fermions is the result of projecting B onto $\mathcal{H}_- = P_- \mathcal{H}$, as the result of introducing the statistics. This projection is $P_- B P_-$, which is equal to $B P_- = P_- B$, whenever B is invariant under the exchange of particles. In this latter case, B leaves \mathcal{H}_- invariant.

The Wigner function of $P_- B P_-$ is $W^{-1}(P_- B P_-) = W^{-1}(P_-) \times W^{-1}(B) \times W^{-1}(P_-)$. Therefore, in order to obtain the function on phase space that describes either an observable or an state of a system of N identical fermions, we must obtain $p_-(u, n) = W^{-1}(P_-)$, the Wigner function of the projection P_- . To do it, one appeals to the linear character of W^{-1} . Thus, $p_-(u, n) = (1/N!) \sum_{\sigma \in S_N} (-1)^\sigma W^{-1}(P_\sigma) W^{-1}(Q_\sigma)$. We recall that the twisted product of two functions on disjoint sets of variables coincides with their ordinary product. Therefore, to obtain the Wigner function of an operator representing a state or observable of a system of N identical fermions, one needs to find the functions $W^{-1}(P_\sigma)$ and $W^{-1}(Q_\sigma)$. Note that $W^{-1}(P_\sigma)$ is defined on the flat phase space $(\mathbb{R}^6)^N$ and $W^{-1}(Q_\sigma)$ on $(S^6)^N$, which makes sense, since P_σ permutes orbital variables and Q_σ spin variables.

To obtain the Wigner function for P_σ one must realize that any permutation of N elements can be written as a product of cyclic permutations or cycles with no common elements. If according to this decomposition, $\sigma = \sigma_1 \dots \sigma_k$, with σ_i a cycle, $i = 1, 2, \dots, k$, we have $P_\sigma = P_{\sigma_1} \dots P_{\sigma_k}$. These operators commute, since each one permutes a different set of variables. Therefore each of the $W^{-1}(P_{\sigma_i})$, $i = 1, 2, \dots, k$ is a function on a different set of variables and, consequently, $W^{-1}(P_\sigma) = W^{-1}(P_{\sigma_1}) W^{-1}(P_{\sigma_2}) \dots W^{-1}(P_{\sigma_k})$. To obtain $W^{-1}(P_\sigma)$, where σ is the cyclic permutation $\sigma = (1, 2, \dots, M)$, we use the Wigner formula

$$\sigma(u_1, u_2, \dots, u_M) = W^{-1}(P_\sigma)(u_1, \dots, u_M) = \int (q - \frac{1}{2}v | P_\sigma | q + \frac{1}{2}v) e^{ip \cdot v} dv \tag{10}$$

with $u_i = (q_i, p_i)$. After some algebra (10) yields

$$\begin{aligned} \sigma(u_1, \dots, u_M) &= 2^{3(M-1)} \exp \left\{ -2i \sum_{k=1; l>k}^M (-1)^{k+l} u_k \mathbf{J} u_l \right\} && M \text{ odd} \\ \sigma(u_1, \dots, u_M) &= (2^{M-1} \pi)^3 \delta(u_1 - u_2 + \dots - u_M) \\ &\times \exp \left\{ -2i \sum_{k=1; l>k}^M (-1)^{k+l} u_k \mathbf{J} u_l \right\} && M \text{ even.} \end{aligned} \tag{11}$$

Here, δ is the Dirac delta and \mathbf{J} has been defined in (7). As an example, if σ is the two cycle which exchanges the variables i and j , one has, after (11),

$$\sigma(u_1, u_2, \dots, u_N) = W^{-1}(P_\sigma) = (2\pi)^3 \delta(u_i - u_j). \tag{12}$$

To evaluate $W^{-1}(Q_\sigma)$, we use the formula $W^{-1}(Q_\sigma) = \text{tr}\{Q_\sigma \Delta(\mathbf{n})\}$, which is obtained from (5) by recalling that Q_σ affects spin variables only and that the Grossmann–Royer operators have trace one (in a generalized sense). This trace can easily be evaluated for any $\sigma \in S_N$ and the final result is

$$W^{-1}(Q_\sigma) = \sum Z_{b_{\sigma(1)}b_1}(\mathbf{n}_1)Z_{b_{\sigma(2)}b_2}(\mathbf{n}_2) \dots Z_{b_{\sigma(N)}b_N}(\mathbf{n}_N) \tag{13}$$

where summation extends to all choices of the $b_i = -$ or $+$.

Now, let H be a function on phase space such that its image by the Weyl mapping, $W(H)$, is an operator representing the Hamiltonian of a system of $N \frac{1}{2}$ -spin particles, without considering the statistics. If we look at them as identical fermions, the true Hamiltonian is $W(H)P_-$ and its Wigner function is $H \times p_-(\mathbf{u}, \mathbf{n})$. We consider this function as the phase-space Hamiltonian of the system. The Moyal propagator associated with this Hamiltonian is $\Xi_-(\mathbf{u}, \mathbf{n}, t) = \Xi(\mathbf{u}, \mathbf{n}, t) \times W^{-1}(P_-)$, where $\Xi(\mathbf{u}, \mathbf{n}, t)$ is the Moyal propagator associated with H . One obtains this formula from the expansion of the propagator in terms of the Hamiltonian. Similarly, we get the formula for the spectral projections as

$$\Gamma_-(\mathbf{u}, \mathbf{n}, E) = (\Gamma \times p_-(\mathbf{u}, \mathbf{n}, E)) \tag{14}$$

where, again, $\Gamma(\mathbf{u}, \mathbf{n}, E)$ are the spectral projections for the Hamiltonian H .

Remark on the notation. The function $p_-(\mathbf{u}, \mathbf{n})$ may depend on the orbital or the spin variables only. In the first case, we denote it as $p_-(\mathbf{u})$ or, more frequently, $p_-(\mathbf{u}_1, \dots, \mathbf{u}_N)$ where N is the number of particles. In the second, $p_-(\mathbf{n})$ as well as $p_-(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_N)$.

2. The Pauli principle

The aim of this section is to present a version of the Pauli principle in the context of the Moyal formalism of quantum mechanics. To begin with, we consider a system formed by two $\frac{1}{2}$ -spins. As already mentioned, the phase space suitable for a description of this situation is $S^2 \times S^2$, the Cartesian product of two two-dimensional spheres. One of our goals is to obtain the function $p_-(\mathbf{n}_1, \mathbf{n}_2) = W^{-1}(P_-)$, $\mathbf{n}_i \in S^2, i = 1, 2$. To achieve it, we have to obtain the functions $\eta_{12}(\mathbf{n}_1, \mathbf{n}_2) = W^{-1}(Q_\sigma)$, where σ is the two-cycle exchanging the labels 1 and 2. After (13), we have that

$$\eta_{12}(\mathbf{n}_1, \mathbf{n}_2) = \sum_{r,s} Z_{rs}(\mathbf{n}_1)Z_{sr}(\mathbf{n}_2) \quad r, s = +, -. \tag{15}$$

The explicit form of the functions Z_{rs} can be found in [5]. They are

$$\begin{aligned} Z_{++}(\mathbf{n}) &= \frac{1}{2}(1 + \sqrt{3} \cos \theta) & Z_{+-}(\mathbf{n}) &= \frac{\sqrt{3}}{2} \sin \theta e^{-i\varphi} \\ Z_{--}(\mathbf{n}) &= \frac{1}{2}(1 - \sqrt{3} \cos \theta) & Z_{-+}(\mathbf{n}) &= \frac{\sqrt{3}}{2} \sin \theta e^{i\varphi}. \end{aligned} \tag{16}$$

Substitution of (16) in (15) gives

$$\eta_{12}(\mathbf{n}_1, \mathbf{n}_2) = \frac{1}{2} + \frac{3}{2}(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)) = \frac{1}{2} + \frac{3}{2}(\mathbf{n}_1 \cdot \mathbf{n}_2). \tag{17}$$

Hence,

$$p_- = \frac{1}{4}(1 - 3\mathbf{n}_1 \cdot \mathbf{n}_2). \tag{18}$$

The next step in our investigation is to find the most general form that a Wigner function can have on the sphere S^2 . Since, in any case, we must have

$$(1 \times f)(\mathbf{n}) = f(\mathbf{n}) \tag{19}$$

if we use the kernel given in (8) for $N = 1$, we obtain

$$f(\mathbf{n}) = A + \mathbf{B} \cdot \mathbf{n} \tag{20}$$

with

$$A = \frac{1}{4\pi} \int_{S^2} f(\mathbf{k}) \cdot d\mathbf{k} \quad B = \frac{3}{4\pi} \int_{S^2} \mathbf{k}[f(\mathbf{k}) \cdot d\mathbf{k}]. \tag{21}$$

The Wigner function of an arbitrary $\frac{1}{2}$ -spin state has been obtained in [5]. Assume that ρ is the density operator corresponding to such a state. Then, there exists a unit vector \mathbf{a} , such that

$$\rho = \alpha|+\rangle\langle +| + \beta|-\rangle\langle -| \quad 0 \leq \alpha, \beta \leq 1 \quad \alpha + \beta = 1 \tag{22}$$

where $|+\rangle$ and $|-\rangle$ are the eigenvectors of $(S \cdot \mathbf{a})$. The Wigner function of ρ is [5]

$$W_\rho(\mathbf{n}) = \frac{1}{2} + \frac{\sqrt{3}}{2}(\alpha - \beta)\mathbf{a} \cdot \mathbf{n}. \tag{23}$$

(23) is easy to obtain from $W_\rho(\mathbf{n}) = \text{tr}[\Delta(\mathbf{n})\rho]$ with the aid of (16). One sees that, in the present case, $A = \frac{1}{2}$ and $|\mathbf{B}| \leq \sqrt{3}/2$. $W_\rho(\mathbf{n})$ represents a pure state if and only if $|\mathbf{B}| = \sqrt{3}/2$. On the other hand, $W_\rho(\mathbf{n})$ is positive if and only if $|\alpha - \beta| \leq \sqrt{3}/3$, which means that no pure state can have a positive Wigner function.

Remark. W considered as a mapping between functions on S^2 of the form given in (20) and operators on \mathbb{C}^2 is one to one. From the considerations in the introduction, we can deduce that if $f(\mathbf{n})$ is such a function, its corresponding operator F can be written as

$$F = \frac{1}{2\pi} \int_{S^2} f(\mathbf{n})\Delta(\mathbf{n}) d\mathbf{n}. \tag{24}$$

However, we could apply (24) to an arbitrary function $f(\mathbf{n})$ on S^2 to obtain an operator on \mathbb{C}^2 . Thus, to any operator F there is a corresponding class of functions. Only one representative of this class is of the form (20) and it is called the Wigner function of F .

Now, let us pick two $\frac{1}{2}$ -spins. Their states are given by the Wigner functions $f(\mathbf{n}_1)$ and $g(\mathbf{n}_2)$. Then, the ensemble is in the state $f(\mathbf{n}_1)g(\mathbf{n}_2)$ which is a function on $S^2 \times S^2$. One may expect that the twisted product of this state by η_{12} on both sides will exchange the roles of both spins. In fact, if we recall that these states must have the form

$$f(\mathbf{n}_1) = A + \mathbf{B} \cdot \mathbf{n}_1 \quad g(\mathbf{n}_2) = C + \mathbf{D} \cdot \mathbf{n}_2 \tag{25}$$

we can prove that

$$\eta_{12} \times f(\mathbf{n}_1)g(\mathbf{n}_2) \times \eta_{12} = f(\mathbf{n}_2)g(\mathbf{n}_1). \tag{26}$$

To prove (26) one has to apply the definition of the twisted product given in section 1 and proceed with the integrals that arise. This calculation is somewhat cumbersome and does not provide any new insight so we omit it. (25) also has an important consequence:

$$p_-(\mathbf{n}_1, \mathbf{n}_2) \times f(\mathbf{n}_1)g(\mathbf{n}_2) \times p_-(\mathbf{n}_1, \mathbf{n}_2) = (AC - \frac{1}{3}(\mathbf{B} \cdot \mathbf{D}))p_-(\mathbf{n}_1, \mathbf{n}_2). \tag{27}$$

One obtains (27) by a direct calculation from (26) using (25).

Since the spins are fermions, (27) represents the true state of the ensemble. When does (27) vanish identically? Obviously, when $\frac{1}{3}\mathbf{B} \cdot \mathbf{D} = AC$. Since $f(\mathbf{n}_1)$ and $g(\mathbf{n}_2)$ are states, they must be of the form of $W_\rho(\mathbf{n})$ in (25). Thus, $AC = \frac{1}{4}$ and $|\mathbf{B}|, |\mathbf{D}| \leq \sqrt{3}/2$. Then, the condition is $\mathbf{B} \cdot \mathbf{D} = \frac{3}{4} = |\mathbf{B}||\mathbf{D}| \cos \omega \implies |\mathbf{B}| = |\mathbf{D}| = \sqrt{3}/2$ and $\omega = 0$. This means that $f(\mathbf{n}_1)$ and $g(\mathbf{n}_2)$ must represent the same pure state. The converse is obvious: if $f(\mathbf{n}_1)$ and $g(\mathbf{n}_2)$ are the Wigner functions for the same pure state, (27) vanishes identically. Consequently, we have obtained a restricted version of the Pauli principle. This result can indeed be improved, as we verify after the following statement.

Proposition. $p_-(\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_N) = 0$ for $N \geq 3$.

Proof. We prove the proposition for $N = 3$ first. To begin with, we write the six permutations of three elements and decompose them as a product of two cycles, which is an easy exercise. After that, the expression for p_- is easy to find:

$$p_-(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \frac{1}{8}\{1 - \eta_{23} - \eta_{12} + \eta_{23} \times \eta_{12} + \eta_{12} \times \eta_{23} - \eta_{13}\}. \tag{28}$$

We want the explicit form of the twisted products in (28). To obtain it, we note that (17) can be written in general as

$$\eta_{ij}(\mathbf{n}_i, \mathbf{n}_j) = \frac{1}{2} + \frac{3}{2}\mathbf{n}_i \cdot \mathbf{n}_j$$

which yields

$$\begin{aligned} (\eta_{23} \times \eta_{12})(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) &= \left(\frac{1}{4\pi}\right)^6 \int d\mathbf{m}_1 d\mathbf{m}_2 d\mathbf{m}_3 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \left(\frac{1}{2} + \frac{3}{2}\mathbf{m}_2 \cdot \mathbf{m}_3\right) \left(\frac{1}{2} + \frac{3}{2}\mathbf{k}_1 \cdot \mathbf{k}_2\right) \\ &\times \prod_{i=1}^3 (1 + 3(\mathbf{n}_i \cdot \mathbf{m}_i + \mathbf{m}_i \cdot \mathbf{k}_i + \mathbf{k}_i \cdot \mathbf{n}_i) + i3\sqrt{3}[\mathbf{n}_i, \mathbf{m}_i, \mathbf{k}_i]). \end{aligned} \tag{29}$$

From (29), one has

$$(\eta_{23} \times \eta_{12})(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \frac{1}{4}\{1 + 3(\mathbf{n}_1 \cdot \mathbf{n}_2 + \mathbf{n}_2 \cdot \mathbf{n}_3 + \mathbf{n}_3 \cdot \mathbf{n}_1) + 3\sqrt{3}i[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]\}. \tag{30}$$

The other twisted product in (28) is

$$\begin{aligned} (\eta_{12} \times \eta_{23}) &= (\eta_{23}^* \times \eta_{12}^*)^* = (\eta_{23} \times \eta_{12})^* \\ &= \frac{1}{4}\{1 + 3(\mathbf{n}_1 \cdot \mathbf{n}_2 + \mathbf{n}_2 \cdot \mathbf{n}_3 + \mathbf{n}_3 \cdot \mathbf{n}_1) - 3\sqrt{3}i[\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3]\} \end{aligned} \tag{31}$$

which finally gives

$$p_-(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = 0. \tag{32}$$

proving our assertion. □

For $N = 4$, one sees that

$$p_-(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4) = \frac{1}{4}\{p_-(\mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4) - \eta_{2134} \times p_-(\mathbf{n}_1, \mathbf{n}_3, \mathbf{n}_4) + \eta_{3124} \times p_-(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_4) - \eta_{4123} \times p_-(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_4)\} = 0 \tag{33}$$

where, for instance, $\eta_{3124} = \eta(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4)$ and this is the Wigner function of the operator that, in the space of spins, produces the permutation

$$\sigma = (3 \ 1 \ 2). \tag{34}$$

One proves the corresponding result for higher values of N by induction.

Remark. It is also true that, in the usual formalism of quantum mechanics, that $P_- = 0$ for systems of three or more spins. To show this for $N = 3$, we write the six 6×6 matrices P_σ to obtain $P_- = 0$. Again, we extend this result to any $N \geq 4$ by induction.

From the proposition, one concludes that the antisymmetrized state function on the phase space of a system of three or more pure spins vanishes. This, along with the consequences of (26), proves the Pauli principle for a system of pure spins.

We can also prove a restricted version of the Pauli principle for $\frac{1}{2}$ -spin systems for which we consider orbital coordinates. To do this, we first state the following lemma:

Lemma. Let σ_{ij} be the permutation that exchanges the variables i and j , and let $\sigma_{ij}(\mathbf{u}_1, \dots, \mathbf{u}_N) = W^{-1}(P_{\sigma_{ij}})$. If $\rho(\mathbf{u}_1, \dots, \mathbf{u}_N)$ is an arbitrary function on \mathbb{R}^{6N} , then

$$(\sigma_{ij} \times \rho \times \sigma_{ij})(\mathbf{u}_1, \dots, \mathbf{u}_i, \dots, \mathbf{u}_j, \dots, \mathbf{u}_N) = \rho(\mathbf{u}_1, \dots, \mathbf{u}_j, \dots, \mathbf{u}_i, \dots, \mathbf{u}_N). \tag{35}$$

Proof. We prove it by a direct calculation using formula (6). We have to note that, in this case, the integral on the angular variables in (6) is equal to one. \square

Now, we state our next result:

Theorem. Assume that in a system of N identical $\frac{1}{2}$ -spin fermions, then

- (i) two of them are in the same quantum state; and
- (ii) these two states are pure and, on the phase space, can be written as a product of a function of the orbital variables multiplied by a function of the spin variables.

Then, the state function of phase space (Wigner function) of the N -particle system vanishes identically.

Proof. We start with a system of two identical $\frac{1}{2}$ -spin particles, both in the same pure state. This means that their respective Wigner functions are identical and equal to $\varphi(\mathbf{u}, \mathbf{n})$, defined on the phase space $(\mathbb{R}^6 \times S^2) \times (\mathbb{R}^6 \times S^2)$. Thus, the Wigner function of the compound state is given by

$$p_- \times \varphi(\mathbf{u}_1, \mathbf{n}_1)\varphi(\mathbf{u}_2, \mathbf{n}_2) \times p_-. \tag{36}$$

Here

$$p_- = p_-(\mathbf{u}_1, \mathbf{u}_2; \mathbf{n}_1, \mathbf{n}_2) = \frac{1}{2}(1 - \sigma(\mathbf{u}_1, \mathbf{u}_2)\eta(\mathbf{n}_1, \mathbf{n}_2)) \tag{37}$$

where $\sigma(\mathbf{u}_1, \mathbf{u}_2) = (2\pi)^3 \delta(\mathbf{u}_1 - \mathbf{u}_2)$ and $\eta(\mathbf{n}_1, \mathbf{n}_2)$ is given by (17). Formula (37) is a consequence of the general formula for $p_-(\mathbf{u}, \mathbf{n})$ given in the introduction.

We claim that (36) is identically zero provided that $\varphi(\mathbf{u}, \mathbf{n})$ can be written as a product of a function on the orbital variables multiplied by another function on the spin variables:

$$\varphi(\mathbf{u}, \mathbf{n}) = \psi(\mathbf{u})\theta(\mathbf{n}). \quad (38)$$

To show this, let us use the decomposition (38) in (36) together with the expression for p_- given by (37). One finds that

$$\begin{aligned} (36) &= \frac{1}{4}(1 - \sigma\eta) \times \psi(\mathbf{u}_1)\theta(\mathbf{n}_1)\psi(\mathbf{u}_2)\theta(\mathbf{n}_2) \times (1 - \sigma\eta) \\ &= \frac{1}{4}\{\psi(\mathbf{u}_1)\psi(\mathbf{u}_2)\theta(\mathbf{n}_1)\theta(\mathbf{n}_2) + (\sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) \times \sigma)(\eta \times \theta(\mathbf{n}_1)\theta(\mathbf{n}_2) \times \eta) \\ &\quad - (\sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2))(\eta \times \theta(\mathbf{n}_1)\theta(\mathbf{n}_2)) - (\psi(\mathbf{u}_1)\psi(\mathbf{u}_2) \times \sigma)(\theta(\mathbf{n}_1)\theta(\mathbf{n}_2) \times \eta)\}. \end{aligned} \quad (39)$$

We recall that the twisted product of two functions on different independent variables is the ordinary product of these functions. Now, consider the second term in (39) first. As a consequence of the lemma and (26) this second term is equal to the first one. To compare the third and fourth terms in (39), one must realize that $\sigma \times \sigma = 1$ and that $\eta \times \eta = 1$. These formulae can be obtained directly from (6). Using them along with the lemma and (26), one finds that

$$\sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) = \sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) \times \sigma \times \sigma = \psi(\mathbf{u}_2)\psi(\mathbf{u}_1) \times \sigma \quad (40a)$$

$$\eta \times \theta(\mathbf{n}_1)\theta(\mathbf{n}_2) = \eta \times \theta(\mathbf{n}_1)\theta(\mathbf{n}_2) \times \eta \times \eta = \theta(\mathbf{n}_2)\theta(\mathbf{n}_1) \times \eta \quad (40b)$$

proving that the third and fourth terms in (39) are equal. Thus,

$$\begin{aligned} p_- \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2)\theta(\mathbf{n}_1)\theta(\mathbf{n}_2) \times p_- \\ = \frac{1}{2}\{\psi(\mathbf{u}_1)\psi(\mathbf{u}_2)\theta(\mathbf{n}_1)\theta(\mathbf{n}_2) - (\sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2))(\eta \times \theta(\mathbf{n}_1)\theta(\mathbf{n}_2))\}. \end{aligned} \quad (41)$$

Next, we use the definition (6) of the twisted product restricted to the orbital variables to find, through a direct calculation, that

$$\begin{aligned} \sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} \psi\left(\frac{\mathbf{u}_1 + \mathbf{u}_2 - \mathbf{x}}{2}\right) \psi\left(\frac{\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{x}}{2}\right) \\ &\quad \times \exp\{i\mathbf{x}\mathbf{J}(\mathbf{u}_2 - \mathbf{u}_1)\} d\mathbf{x}. \end{aligned} \quad (42)$$

Now, assume that $\psi(\mathbf{u})\theta(\mathbf{n})$ is the Wigner function of a pure state. In the language of the Wigner functions this is equivalent to saying that $\psi(\mathbf{u})\theta(\mathbf{n}) = \psi(\mathbf{u})\theta(\mathbf{n}) \times \psi(\mathbf{u})\theta(\mathbf{n}) = (\psi(\mathbf{u}) \times \psi(\mathbf{u}))(\theta(\mathbf{n}) \times \theta(\mathbf{n}))$. This implies that

$$\psi(\mathbf{u}) \times \psi(\mathbf{u}) = \psi(\mathbf{u}) \quad \text{and} \quad \theta(\mathbf{n}) \times \theta(\mathbf{n}) = \theta(\mathbf{n}). \quad (43)$$

Thus, $\psi(\mathbf{u})$ and $\theta(\mathbf{n})$ are pure states on the orbital and spin variables respectively. Our next goal is to show that, in this case, one finds that

$$\sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) = \psi(\mathbf{u}_1)\psi(\mathbf{u}_2). \quad (44)$$

To prove (44), $\psi(u)$ must be a Wigner function of a pure quantum state, i.e. there must exist a vector $\Psi(x)$ in the Hilbert space of the single-particle states such that $\psi(u)$ is the Wigner function of $\Psi(x)$. On the other hand, (44) is not true, in general, for an arbitrary function $h(u)$ on the flat phase space. Take, for instance, $\psi(u) \equiv 1$. Then, $\sigma \times 1 = \sigma \neq 1$. In fact, the function identical to one is the Wigner function of the identity operator which is not even a density operator, since it has infinite trace.

To prove (44), we re-interpret a formula from O'Connell and Wigner [11]. Let $P(q_1, p_1, q_2, p_2)$ be the Wigner function of the state of two identical bosons. Then,

$$\begin{aligned}
 &P(q + q_2, p + p_2, q - q_2, p - p_2) \\
 &= \left(\frac{2}{\pi}\right)^3 \int P(q + q_1, p + p_1, q - q_1, p - p_1) e^{4i(p_1 \cdot q_2 - q_1 \cdot p_2)} dq_1 dp_1 \quad (45)
 \end{aligned}$$

where q_i and p_i are, respectively, the position and momentum of the i th boson ($i = 1, 2$). Although O'Connell and Wigner [11] prove this formula for bosons, it can be used in our case because of the arguments given thereafter.

O'Connell and Wigner obtain (45) as follows [11]. Assume that $\Psi(x_1, x_2)$ is the wavefunction (in Hilbert space) representing the state of two identical bosons. Then, $\Psi(x_1, x_2) = \Psi(x_2, x_1)$ and its corresponding density matrix

$$\rho(x_1, x_2, x'_1, x'_2) = \Psi^*(x_1, x_2)\Psi(x'_1, x'_2) \quad (46)$$

has the following property

$$\rho(x_1, x_2, x'_1, x'_2) = \rho(x_1, x_2, x'_2, x'_1). \quad (47)$$

From (47), the authors obtain (45) by means of linear transformations such as integrations and Fourier transforms.

In our case, let $\Psi(x)$ be the state in the Hilbert space whose Wigner function is $\psi(u)^\dagger$. Then, the vector $\Psi(x_1, x_2) = \Psi(x_1) \otimes \Psi(x_2) = \Psi(x_1)\Psi(x_2)$ admits $\psi(u_1)\psi(u_2)$ as a Wigner function, since the variables corresponding to particles 1 and 2 are independent. Therefore, the density matrix of this two-particle state has the following property:

$$\begin{aligned}
 \rho(x_1, x_2, x'_1, x'_2) &= \Psi^*(x_1)\Psi^*(x_2)\Psi(x'_1)\Psi(x'_2) \\
 &= \Psi^*(x_1)\Psi^*(x_2)\Psi(x'_2)\Psi(x'_1) = \rho(x_1, x_2, x'_2, x'_1). \quad (48)
 \end{aligned}$$

This is the point of departure to arrive to (45). Therefore, if $\psi(u_1)\psi(u_2) = P(q_1, p_1, q_2, p_2)$, by performing the same type of manipulation as in [11], we obtain (45), which remains valid in our case.

To apply (45) here (with $P(q_1, p_1, q_2, p_2) = \psi(u_1)\psi(u_2)$, $u_i = (q_i, p_i)$, $i = 1, 2$), we first perform the following change of variables

$$\frac{1}{2}(u_1 + u_2) = (q, p) \quad \frac{1}{2}x = (q_1, p_1) \quad \frac{1}{2}(u_2 - u_1) = -(q_2, p_2). \quad (49)$$

Thus, the integral in (42) becomes

$$2^6 \int \psi(q - q_1, p - p_1)\psi(q + q_1, p + p_1) \exp\{4i(p_1 \cdot q_2 - q_1 \cdot p_2)\} dq_1 dp_1. \quad (50)$$

† The Wigner function of a vector state is the Wigner function of the density operator associated with this vector.

Thus (45) gives

$$\begin{aligned} & \psi(\mathbf{q} + \mathbf{q}_2, \mathbf{p} + \mathbf{p}_2)\psi(\mathbf{q} - \mathbf{q}_2, \mathbf{p} - \mathbf{p}_2) \\ &= \left(\frac{2}{\pi}\right)^3 \int \psi(\mathbf{q} + \mathbf{q}_1, \mathbf{p} + \mathbf{p}_1)\psi(\mathbf{q} - \mathbf{q}_1, \mathbf{p} - \mathbf{p}_1)e^{4i(\mathbf{p}_1 \cdot \mathbf{q}_2 - \mathbf{q}_1 \cdot \mathbf{p}_2)} d\mathbf{q}_1 d\mathbf{p}_1 \quad (45a) \end{aligned}$$

and it becomes evident that the right-hand side of (45a) is exactly the right-hand side of (42) and that

$$\sigma \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) = \psi(\mathbf{u}_1)\psi(\mathbf{u}_2) \quad (51)$$

provided that $\psi(\mathbf{u})$ is the Wigner function of a pure quantum state.

Thus, (51) and (41) give

$$\begin{aligned} & p_- \times \psi(\mathbf{u}_1)\psi(\mathbf{u}_2)\theta(\mathbf{n}_1)\theta(\mathbf{n}_2) \times p_- \\ &= \frac{1}{2}\{\theta(\mathbf{n}_1)\theta(\mathbf{n}_2) - \eta \times \theta(\mathbf{n}_1)\theta(\mathbf{n}_2)\}\{\psi(\mathbf{u}_1)\psi(\mathbf{u}_2)\} \\ &= \psi(\mathbf{u}_1)\psi(\mathbf{u}_2)\left(\frac{1}{2} - \frac{1}{3}|B|^2\right)p_-(\mathbf{n}_1, \mathbf{n}_2) \quad (52) \end{aligned}$$

where $\theta(\mathbf{n}) = \frac{1}{2} + \mathbf{B} \cdot \mathbf{n}$ (see (20) and (23)) and $p_-(\mathbf{n}_1, \mathbf{n}_2)$ is as in (18) (see (27)). We have already proven that if $\theta(\mathbf{n})$ is a pure spin state, then $|B| = \sqrt{3}/2$ which implies that the last formula in (52) vanishes. This proves our claim if the number of particles is equal to two.

Now, let us consider a system of three identical $\frac{1}{2}$ -spin fermions. Let $\varphi_1(\alpha_1)$, $\varphi_2(\alpha_2)$ and $\varphi_3(\alpha_3)$ be the Wigner functions representing the respective individual states on phase space ($\alpha_i = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{n}_i)$ $i = 1, 2, 3$). The Wigner function of the compound system is then

$$p_-(\alpha_1, \alpha_2, \alpha_3) \times \varphi_1(\alpha_1)\varphi_2(\alpha_2)\varphi_3(\alpha_3) \times p_-(\alpha_1, \alpha_2, \alpha_3) \quad (53)$$

where

$$p_-(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{3!}(1 - p_{213} - p_{132} - p_{321} + p_{312} + p_{231}). \quad (54)$$

Here, $p_{\alpha\beta\gamma}(\alpha_1, \alpha_2, \alpha_3)$ is the Wigner function of the operator realizing the permutation ($\alpha\beta\gamma$) of the wavefunction of a three-particle state.

After (54), we can write (53) as

$$\begin{aligned} & \frac{1}{3}(1 + p_{312} + p_{231}) \times \frac{1}{2}(1 - p_{132}) \times \varphi_1(\alpha_1)\varphi_2(\alpha_2)\varphi_3(\alpha_3) \\ & \times \frac{1}{2}(1 - p_{132}) \times \frac{1}{3}(1 + p_{312} + p_{231}). \quad (55) \end{aligned}$$

Assume now that $\varphi_2 \equiv \varphi_3$ and that these states

(i) can be written as a product of a function of the orbital variables multiplied by a function of the spin variables; and

(ii) represent Wigner functions of a pure state.

Since p_{132} permutes 2 and 3, it is a function of α_2 and α_3 only. Therefore,

$$\begin{aligned} & \frac{1}{2}(1 - p_{132}) \times \varphi_1(\alpha_1)\varphi_2(\alpha_2)\varphi_3(\alpha_3) \times \frac{1}{2}(1 - p_{132}) \\ &= \varphi_1(\alpha_1)\left[\frac{1}{2}(1 - p_{132}) \times \varphi_2(\alpha_2)\varphi_3(\alpha_3) \times \frac{1}{2}(1 - p_{132})\right]. \quad (56) \end{aligned}$$

The term between brackets in (56) is of the form (36) with (38) and is, therefore, equal to zero. This proves our assertion when the number of particles N is equal to three.

Remark. Clearly, there is no loss of generality in imposing these conditions to the labels 2 and 3. The same result is obtained using 1 and 2 as well as 1 and 3. We also observe that no restriction should be imposed on the third state (in our calculation $\varphi_1(\alpha_1)$).

For $N > 3$ the Wigner function of the system is given by

$$p_- \times \varphi_1(\alpha_1)\varphi_2(\alpha_2) \dots \varphi_N(\alpha_N) \times p_- \quad (57)$$

If $\varphi_1 \equiv \varphi_2$, we can prove that (57) vanishes identically. The proof is identical to the one presented here for the case $N = 3$ and, therefore, we omit it. We wish to note that no extra conditions are imposed on the Wigner functions $\varphi_3(\alpha_3) \dots \varphi_N(\alpha_N)$. This proves our theorem. \square

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